Graphing Sine and Cosine Functions

Any equation of the form $y = a\sin\left[b\left(x-c\right)\right]+d$ with $a \neq 0$ and $b \neq 0$ is a **sine function**. Its graph is called a **sine wave**, **sinusoidal wave**, or **sinusoid.** The graph of any sine function is a transformation of the graph of $y = \sin x$.

★ Assume x is in radians unless the problem specifically states that it is in degrees!

As the terminal side of an angle rotates around the unit circle, how does the value of the sine change?

- From 0 to $\pi/2$, the sine _______________ from _____ to _____.
- From $\pi/2$ to π, the sine _______________ from _____ to _____.
- From π to $3\pi/2$, the sine _______________ from _____ to _____.
- From $3\pi/2$ to 2π, the sine _______________ from _____ to _____.
- The cycle repeats.

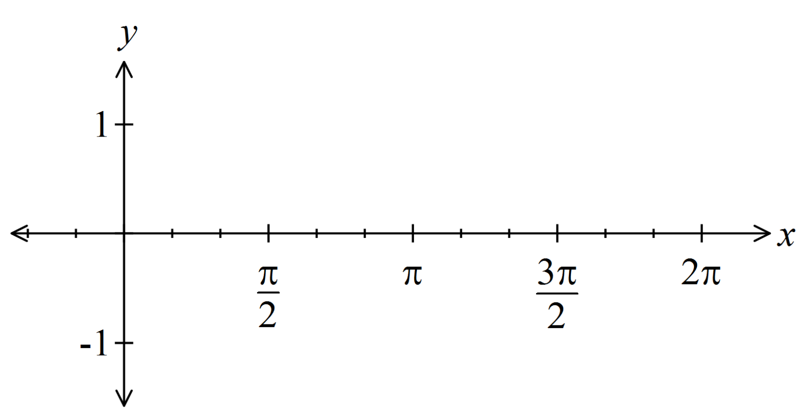
Because $\sin\left(x+2\pi\right)=\sin x$, the shape we see in the interval $\left[0,2\pi\right]$ repeats on the intervals $\left[2\pi,4\pi\right]$, $\left[4\pi,6\pi\right]$, $\left[-2\pi,0\right]$, $\left[-4\pi,-2\pi\right]$, etc.

A repeating function like $y=\sin x$ is called a **periodic function.** The length of the smallest non-repeating unit is the **period** of the function. The period of $y=\sin x$ is 2π. The graph of $y=\sin x$ over any interval of length 2π is called a **cycle.**

The graph of $y=\sin x$ over $\left[0,2\pi\right]$ is the **fundamental cycle.**

Key points on the graph of $y=\sin x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y=\sin x$					



As the terminal side of an angle rotates around the unit circle, how does the value of the cosine change?

- From 0 to $\pi/2$, the cosine ________________ from _____ to _____.
- From $\pi/2$ to π, the cosine ________________ from _____ to _____.
- From π to $3\pi/2$, the cosine ________________ from _____ to _____.
- From $3\pi/2$ to 2π, the cosine ________________ from _____ to _____.
- The cycle repeats.

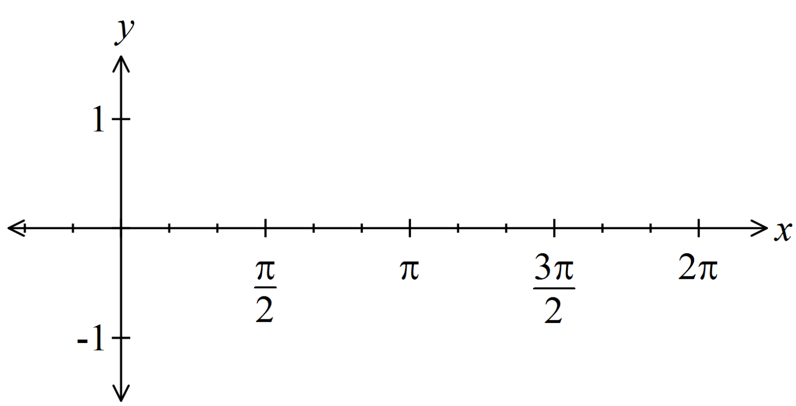
Because $\cos(x+2\pi)=\cos x$, the shape we see in the interval $[0,2\pi]$ repeats on the intervals $[2\pi,4\pi]$, $[4\pi,6\pi]$, $[-2\pi,0]$, $[-4\pi,-2\pi]$, etc.

The graph of $y=\cos x$ has the same shape as the graph of $y=\sin x$, but it is shifted to the left by a distance of $\pi/2$. For this reason, the graph of $y=\cos x$ is also called a sine wave.

The graph of $y=\cos x$ over $[0,2\pi]$ is called the **fundamental cycle** of $y=\cos x$.

Key points on the graph of $y=\cos x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y=\cos x$					

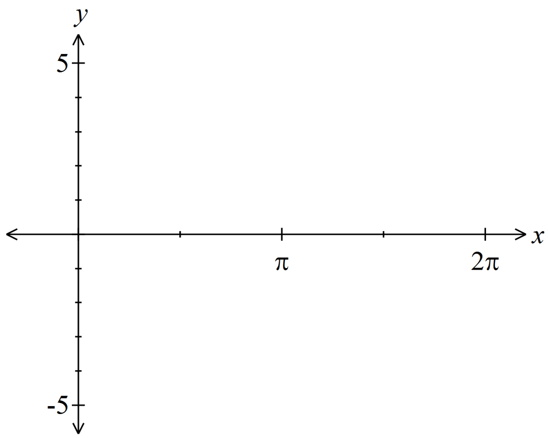


The effect of changing the value of *a*:

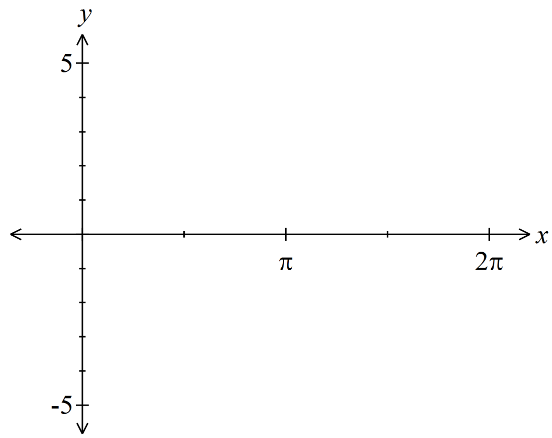
The **amplitude** of $y=a\sin x$ or $y=a\cos x$ is $|a|$. The amplitude is the "height" of the sine wave. It is half the difference between the maximum and minimum points on the graph. If a is negative, the graph is reflected over the *x*-axis.

Examples: Sketch the graphs of the following and determine the amplitude and range of each.

$y = 3\sin x$



$y = -5\cos x$

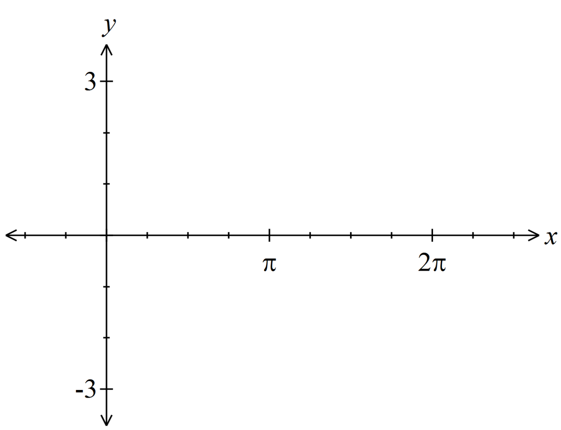


The effect of changing the value of *c*:

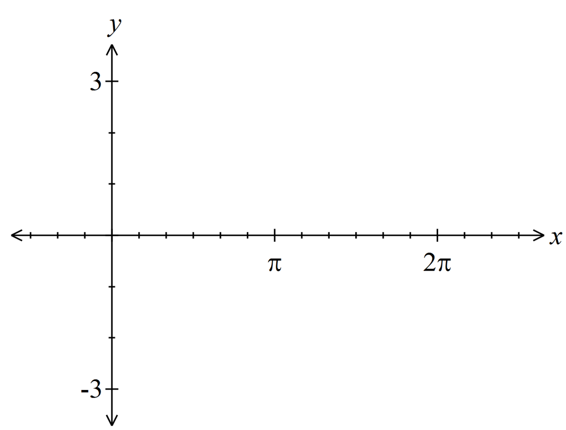
The **phase shift** of the graph of $y = \sin(x - c)$ or $y = \cos(x - c)$ is c. Notice that the sign of c is the opposite of the sign in the equation. This means that the graph is shifted c units to the right if c is positive, or c units to the left if c is negative.

Examples: Sketch each graph and find the amplitude, phase shift, and range of each function.

$y = \cos(x - \pi/4)$



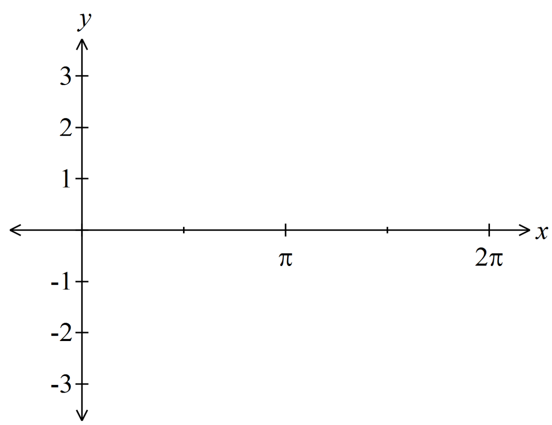
$y = 2\sin(x + \pi/3)$



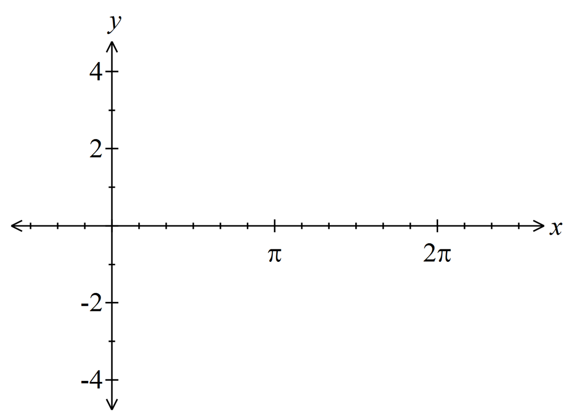
The effect of changing the value of *d*:
The **vertical translation** of the graph of $y = \sin x + d$ or $y = \cos x + d$ is d. This means that the graph is shifted *d* units up if *d* is positive, or *d* units down if *d* is negative.

Examples: Sketch each graph and find the amplitude, phase shift, vertical shift, and range of each function

$y = \sin x + 2$



$y = 3\cos(x - \pi/6) - 1$



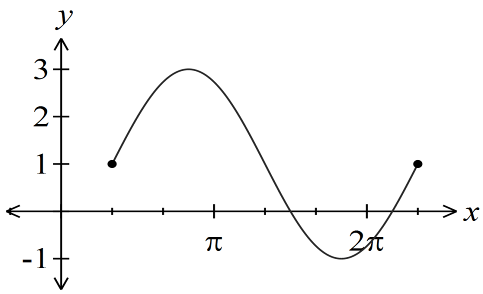
Examples: Find the equation of each sine wave in its final position.

1. The graph of $y = \sin x$ is stretched by a factor of 2, reflected in the *x*-axis, shifted $\pi/5$ units to the right, then translated 4 units downward.

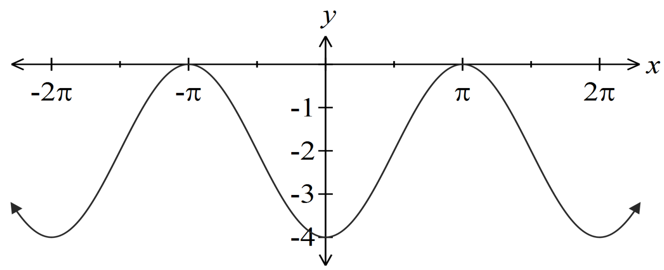
2. The graph of $y = \cos x$ is shifted $\pi/3$ units to the left, translated upward 2 units, then stretched by a factor of 2.

Examples: Find an equation of the requested form whose graph is the given sine wave.

$y = a\sin(x-c)+d$



$y = a\cos(x-c)+d$

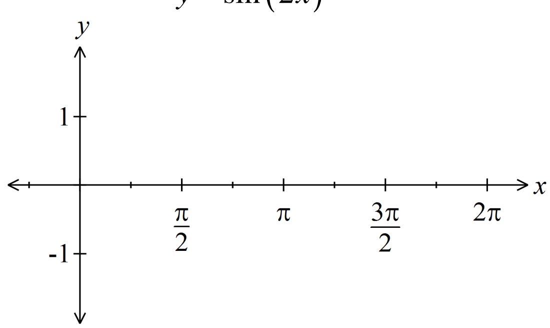


The effect of changing the value of *b*:

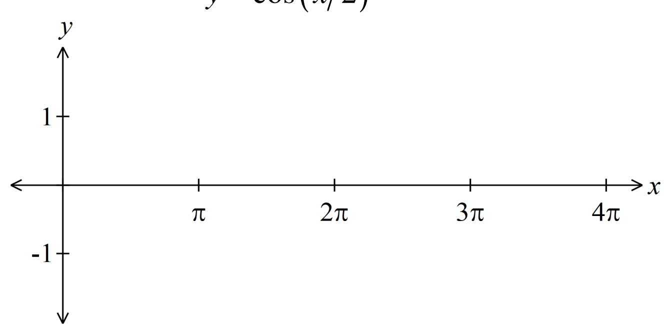
The **period** of the graph of $y = \sin(bx)$ or $y = \cos(bx)$ for $b > 0$ is $P = 2\pi/b$. This means that there are b cycles every 2π units. The **frequency**, F, of a sine wave with period P is defined by $F = 1/P = b/2\pi$.

Examples: Sketch the graphs of the following and determine the period and frequency of each.

$y = \sin(2x)$



$y = \cos(x/2)$



The general sine wave:

Characteristics of the graph of $y = a\sin\left[b\left(x-c\right)\right]+d$ or $y = a\cos\left[b\left(x-c\right)\right]+d$:

- Amplitude: $|a|$
- Period: $P = 2\pi/b$
- Frequency: $F = 1/P = b/2\pi$
- Phase shift: c (Remember that the sign of c is the opposite of the sign in the equation).
  - Shift right for $c > 0$.
  - Shift left for $c < 0$.
- Vertical translation: d
  - Shift up for $d > 0$.
  - Shift down for $d < 0$.

Steps to graph $y = a\sin\left[b\left(x-c\right)\right]+d$ or $y = a\cos\left[b\left(x-c\right)\right]+d$:

Start with the five key points on the graph of $y = \sin x$ or $y = \cos x$.

Find five key points for $y = a\sin\left[b\left(x-c\right)\right]+d$ or $y = a\cos\left[b\left(x-c\right)\right]+d$:

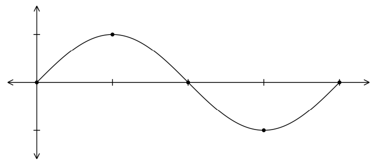
1. Divide each x-coordinate by b (adjusts period) and add c (phase shift).
2. Multiply each y-coordinate by a (adjusts amplitude and, if necessary, reflects graph over x-axis) and add d (vertical shift).
3. Sketch one cycle of your graph through the five new points.

***Note:** Order is important. Multiply or divide first (stretches/compressions), then add (shifts).

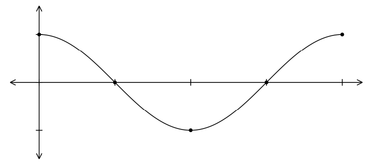
For those who prefer to do things visually:

1. Determine the vertical shift and draw the midline.
2. Determine the amplitude and draw dashed lines across the maximum and minimum values.
3. Find the new x-coordinates:
   a. Use the phase shift c to determine where the cycle begins.
   b. Move one period to the right to determine where the cycle ends.
   c. Divide the period by 4 to determine how far apart the key x-coordinates are.
   d. Determine the coordinates of the key x-coordinates.
4. Duplicate the shape of the fundamental cycle of the parent graph through the new x-coordinates. (Remember to flip it upside-down if a is negative.)

Sine: middle-top-middle-bottom-middle



Cosine: top-middle-bottom-middle-top



★ **Note:** Feel free to combine the two methods. Personally, I prefer to deal with vertical adjustments (amplitude and vertical shift) visually, and adjust the x-coordinates using key points.

Examples: Determine the amplitude, period, frequency, phase shift, and vertical shift of the following. Then sketch one cycle of each graph. Draw and label your own axes.

a) $y = \sin\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] + 1$

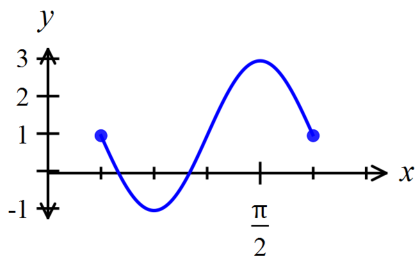
b) $y = -2\cos\left(2x + \frac{\pi}{2}\right) - 2$

c) $y = 3\cos\left[4\left(x - \frac{\pi}{6}\right)\right] - 1$

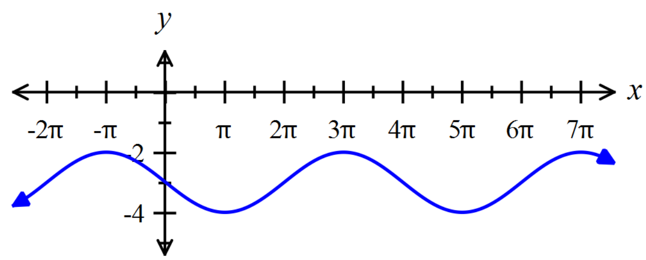
d) $y = -4\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}\right)$

Examples: Find an equation of the requested form whose graph is the given sine wave.

$y = a\sin\left[b\left(x-c\right)\right]+d$



$y = a\cos\left[b\left(x-c\right)\right]+d$



Graphing Secant and Cosecant Functions

Remember, $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$.

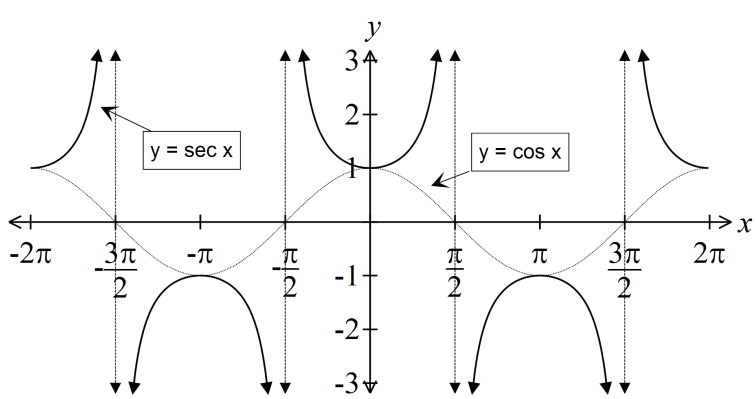
- We aren't allowed to divide by 0. This means:
  - Whenever $\cos x = 0$, $\sec x$ is undefined, and whenever $\sin x = 0$, $\csc x$ is undefined.
    - Places where $\cos x = 0$ and $\sec x$ is undefined:______________________________
    - Places where $\sin x = 0$ and $\csc x$ is undefined:______________________________
  - The graphs of $y = \sec x$ and $y = \csc x$ have vertical asymptotes at these locations.
  - **To Find the Equations of the Asymptotes:**
    - Start with any x-value where the function is undefined.
      Add this value to k times the distance between the asymptotes.
      $x = \text{asymptote} + (\text{distance between asymptotes}) \cdot k$

Graphing Secant Functions

- To graph $y = a\sec\left[b\left(x-c\right)\right]+d$:
  - Sketch the graph of $y = a\cos\left[b\left(x-c\right)\right]+d$.
  - Wherever the graph of the cosine function crosses its center point, draw a vertical asymptote.
  - The local maxima of the graph of the cosine function become local minima on the graph of the secant function with $y \to \infty$ as x approaches the asymptotes on either side. The local minima of the graph of the cosine function become local maxima on the graph of the secant function with $y \to -\infty$ as x approaches the asymptotes on either side.

Key points on the graph of $y = \sec x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \sec x$	1	undef.	-1	undef.	1

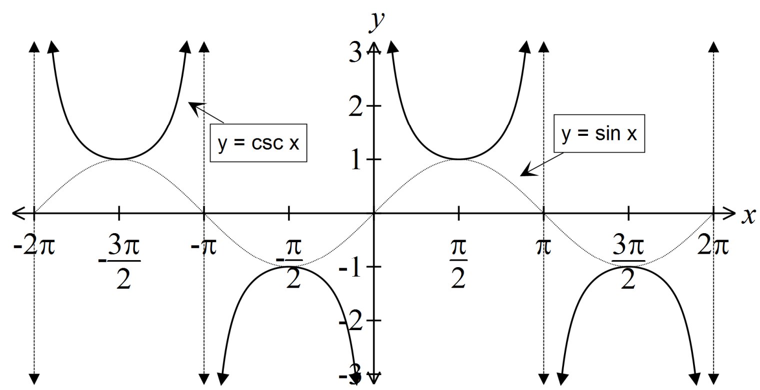


Graphing Cosecant Functions

- To graph $y = a\csc\left[b(x-c)\right]+d$:
  - Sketch the graph of $y = a\sin\left[b(x-c)\right]+d$.
  - Wherever the graph of the sine function crosses its center point, draw a vertical asymptote.
  - The local maxima of the graph of the sine function become local minima on the graph of the cosecant function with $y \to \infty$ as x approaches the asymptotes on either side. The local minima of the graph of the sine function become local maxima on the graph of the cosecant function with $y \to -\infty$ as x approaches the asymptotes on either side.

Key points on the graph of $y = \csc x$:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \csc x$	undef.	1	undef.	-1	undef.



Examples: Graph the following functions. Find the period, asymptotes, and range of each.

a) $y = 3\sec(2x)$

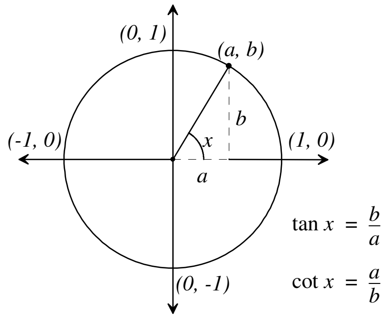
b) $y = \csc\left(x - \dfrac{\pi}{4}\right) + 2$

c) $y = \sec\left(\dfrac{1}{2}x + \dfrac{\pi}{6}\right)$

d) $y = 2\csc\left(\dfrac{\pi}{4}x + \dfrac{3\pi}{4}\right)$

Graphing Tangent and Cotangent Functions

Let (a,b) be coordinates of points on the unit circle. For any given angle x, $\tan x = b/a$. This means that $y = \tan x$ is undefined whenever $a = 0$. For any given angle x, $\cot x = a/b$. This means that $y = \cot x$ is undefined whenever $b = 0$. Notice that it takes π radians for the values of the tangent and cotangent to make one complete cycle.

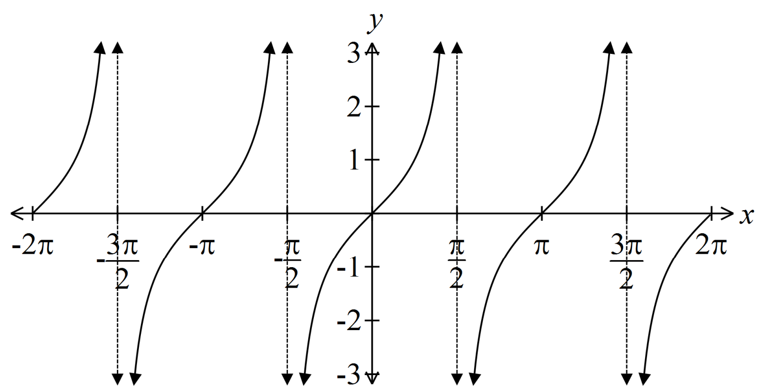


Graphing Tangent Functions:

The domain of $y = \tan x$ is the set of all real numbers except numbers of the form $\pi/2 + k\pi$, where k is an integer. The equations of the vertical asymptotes are $x = \pi/2 + k\pi$, where k is an integer.

Key points on the graph of $y = \tan x$:

x	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
$y = \tan x$	undef.	-1	0	1	undef.



To graph $y = a\tan\left[b(x-c)\right] + d$:

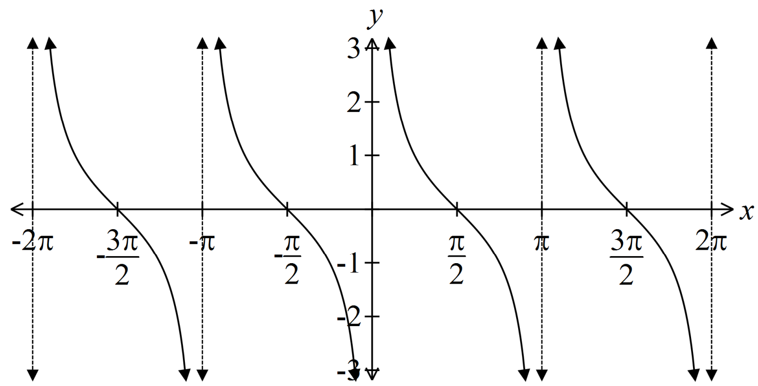
1. Start with the three key points on the graph of $y = \tan x$ and the equations of the asymptotes.
2. Find three key points and the asymptotes for $y = a\tan\left[b(x-c)\right] + d$ by:
    a. dividing each x-coordinate by b and adding c. (Treat the equations of the asymptotes like x-coordinates.)
    b. multiplying each y-coordinate by a and adding d.
3. Sketch one cycle of $y = a\tan\left[b(x-c)\right] + d$ through the three new points and approaching the new asymptotes.

★ **The period of $y = a\tan\left[b(x-c)\right] + d$ and $y = a\cot\left[b(x-c)\right] + d$ is π/b rather than $2\pi/b$.**

Graphing Cotangent Functions:
The domain of $y = \cot x$ is the set of all real numbers except numbers of the form $k\pi$, where k is an integer.
The equations of the vertical asymptotes are $x = k\pi$, where k is an integer.

Key points on the graph of $y = \cot x$:

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$y = \cot x$	undef.	1	0	-1	undef.



To graph $y = a\cot\left[b\left(x-c\right)\right]+d$:

1. Start with the three key points on the graph of $y = \cot x$ and the equations of the asymptotes.
2. Find three key points and the asymptotes for $y = a\cot\left[b\left(x-c\right)\right]+d$ by:
   a. dividing each x-coordinate by b and adding c. (Treat the equations of the asymptotes like x-coordinates.)
   b. multiplying each y-coordinate by a and adding d.
3. Sketch one cycle of $y = a\cot\left[b\left(x-c\right)\right]+d$ through the three new points and approaching the new asymptotes.

Examples: Graph the following functions. Find the period and the equations of the asymptotes of each.

$y = \tan\left(\frac{1}{2}x\right)$

$y = \frac{1}{2}\cot\left(x+\frac{\pi}{3}\right)$

$y = 3\tan\left(2x+\frac{\pi}{2}\right)+1$

$y = 2\cot\left[3\left(x-\frac{\pi}{6}\right)\right]-1$